I. INTRODUCTION

Global optimization is a subject of intense current interest, both scientific and commercial. For example, improved solutions to optimization problems, such as the traveling-salesman problem and the routing of circuitry on a chip, could lead to cost reductions and improved performance. In the chemistry and physics communities motivation is often provided by the structural insight which can be derived from finding the lowest energy configuration of a (macro)molecular system. Therefore, the development of better global optimization algorithms is an important task. It is a challenge which needs to be guided by theoretical insight rather than proceeding on purely empirical or intuitive grounds.

The main difficulty associated with global optimization is the exponential increase in the search space with system size. One expression of this problem is Levinthal’s paradox which points out the apparent impossibility of finding the native state of a protein. The number of possible conformations of a typical protein is so large that it would take longer than the age of the universe to find the native state if the conformations were sampled randomly, even at an unfeasibly rapid rate. This problem can be more rigorously defined using computational complexity theory. Finding the global minimum of a protein or a cluster is “NP-hard,” a class of problems for which there is no known algorithm that is guaranteed to find a solution in polynomial time.

However, we know that proteins are able to reach their native state relatively rapidly. The resolution of Levinthal’s paradox lies in the fact that the search is not random—the paradox results from the implicit assumption that the potential energy surface (PES) is flat. In reality, the topography of the PES is crucial in determining the ability of a system to reach the global minimum. For example, the system is thermodynamically more likely (than random) to be in those regions of the PES that are low in energy, and downhill pathways can guide the system towards certain configurations. A funnel, a set of downhill pathways that converge on a single low energy minimum, combines these two effects. It has been suggested that the PES’s of proteins are characterized by a single deep funnel and that this feature underlies the ability of proteins to fold to their native state. Indeed it is easy to design model single-funnel PES’s that result in efficient relaxation to the global minimum, despite very large configurational spaces.

Therefore, global optimization should be relatively easy for those systems with a single-funnel PES topography. However, there are many systems for which the topography of the PES does not permit escape from the huge number of disordered configurations. These systems are more likely to form glassy or amorphous structures on cooling. In such cases it is very difficult for global optimization to succeed. This is especially true if the optimization method, e.g., simulated annealing, tries to mimic some physical process, because the natural time scales for the formation of order are much longer than those that can be probed by computer simulation.

It has been suggested that the PES’s of such “glassy” materials are rough, with many funnels leading to different low energy amorphous configurations, rather than to the crystal. However, one does not need to have much complexity to make global optimization difficult; two funnels are enough if the system is more likely to enter the secondary funnel on descending the PES. In this paper we examine one such example from the realm of atomic clusters and compare it with another much easier problem. Our aims are to pinpoint why many global optimization algorithms are likely to fail for this cluster, to understand the reasons for the success of the “basin-hopping” method and to deduce lessons for the design of global optimization algorithms. In particular, we highlight the influence of the thermodynamics of the clusters on these questions. A brief description of some of the results has already appeared.
The Marks decahedral global minima have been found by a number of exceptions. The LJ 38 global minimum is a face-centered cubic (fcc) truncated octahedron [Fig. 1(a)], and for 75–77 and 102–104 atoms the global minima are based on Marks decahedra.28 These clusters are interesting because the global minima are much more difficult to find by an unbiased global optimization method. They were all initially discovered by construction using physical insight.20,21 Since then the truncated octahedron has been found by a number of methods.19,22–24,29,30 The Marks decahedral global minima have only been found by the basin-hopping approach,23,31 the method we analyze in this paper, and a modified genetic algorithm which searches the same transformed surface.32

For the above reasons we choose to examine in more detail the behavior of LJ38, and as a contrast LJ55. The LJ55 global minimum is a complete Mackay icosahedron [Fig. 1(c)], a structure that is easily found by any reasonable global optimization algorithm.

For LJ38 the second lowest energy minimum is an incomplete Mackay icosahedron with C5v point group symmetry [Fig. 1(b)], which lies only 0.676\(\epsilon\) higher in energy than the fcc global minimum. As might be expected from their structural dissimilarity, the two lowest energy minima are well separated on the PES. Using a method which walks over the PES from minimum to minimum via transition states,33–35 we have been able to find paths connecting these two structures. The lowest energy path that we found is depicted in Fig. 2.36 It clearly shows that there is a large activation energy associated with passage between the two lowest energy minima. The fcc and icosahedral regions of configuration space represent two distinct funnels on the PES. The minima at the top of the barrier are from the region of configuration space associated with the liquid-like state (Fig. 3). At temperatures below the melting point the Boltzmann weights of these states are small implying that there is also a large free energy barrier between the two funnels.37 The dynamics confirm this picture; below the melting point a simulation run started in one of the funnels always remains trapped in that funnel.

The LJ55 PES is very different. All the low energy minima for LJ55 are based on the Mackay icosahedron. The first three peaks above the global minimum in the probability distribution (Fig. 3) are associated with the liquid-like state. These icosahedral minima lower in energy than the second lowest energy fcc minimum (Fig. 3). The fcc funnel is narrow, whereas the icosahedral funnel is wider and has a broad, fairly flat bottom.
distribution of minima in Fig. 4 represent Mackay icosahedra with one, two and three surface defects, respectively. Furthermore, the Mackay icosahedron is 2.644 eV absolute of the superposition method is that it can be used to calculate thermodynamics. Methods such as multihistogram Monte Carlo, often the method of choice for clusters, are not able to describe the low temperature thermodynamics for LJ38 correctly because in simulations the cluster is not able to cross the large free energy barrier between the fcc and icosahedral funnels. It might be possible to overcome this difficulty by using more advanced techniques such as jump-walking or umbrella sampling, however, these methods can be computationally demanding. The advantage of the superposition method is that it can be used to calculate absolute densities of states for different regions of phase space, and therefore there is no need to cross any free energy barrier. Second, the superposition method can give greater physical insight into the relationship between the topography of the PES and the thermodynamics. We review the superposition method here because it will be important later when we formulate the thermodynamic properties of a transformed energy surface, and because some additional developments need to be made before it can be applied to LJ55. The fundamental relation is

\[ \Omega(E) = \sum_{E_i \leq E} n_s \Omega_s(E) \]

where the sum is over all the geometrically distinct minima on the PES, \( \Omega_s(E) \) is the density of states of a single minimum \( s \), \( E \) is the potential energy of minimum \( s \), and \( n_s \), the number of permutational isomers of minimum \( s \), is given by \( n_s = 2N!/(h_s^s) \), where \( h_s \) is the order of the point group of \( s \). The equation is exact and just expresses the division of configuration space into basins of attraction that surround each minimum on the PES. One difficulty with the above equation is that \( \Omega_s(E) \) is not known \textit{a priori}. However, if the basin is assumed to be harmonic then the form of \( \Omega_s(E) \) is simple, and using this expression a qualitative picture of the thermodynamics can be obtained. Furthermore, anharmonic expressions for \( \Omega_s(E) \) have also been derived which are able to describe the thermodynamics of Lennard-Jones clusters more accurately. The second difficulty with Eq. (2) is that for all but the very smallest clusters the sum involves an impractically large number of minima. Hoare and McInnes and more recently Tsai and Jordan have enumerated lower bounds to the number of geometric isomers for LJ clusters from 6 to 13 atoms. This number rises exponentially with \( N \). Extrapolating the trend gives for LJ55 an estimate of 10^{32} geometric isomers. In such a case, as it is not possible to obtain a complete set of minima, one instead has to use a representative sample, which, for example, could be obtained by performing minimizations from a large set of configurations generated by a molecular dynamics trajectory. However, one needs a way to correct for the incomplete nature of the sample. This correction can be achieved by weighting the density of states for each known minimum by \( g_s \), the number of minima of energy \( \sim E \), for which the minimum \( s \) is representative. Hence,

\[ \Omega(E) = \sum_{E_i \leq E} g_s n_s \Omega_s(E) \]

IV. THERMODYNAMICS OF LJ38 AND LJ55

To calculate the thermodynamic properties of our clusters we use the superposition method in this approach the density of states, \( \Omega(E) \), is constructed using information about a set of minima on the PES (such as those represented in Figs. 3 and 4). This choice is motivated by two considerations. First, methods such as multihistogram Monte Carlo, often the method of choice for clusters, are not able to describe the low temperature thermodynamics for LJ38 correctly because in simulations the cluster is not able to cross the large free energy barrier between the fcc and icosahedral funnels. It might be possible to overcome this difficulty by using more advanced techniques such as jump-walking or umbrella sampling, however, these methods can be computationally demanding. The advantage of the superposition method is that it can be used to calculate absolute densities of states for different regions of phase space, and therefore there is no need to cross any free energy barrier. Second, the superposition method can give greater physical insight into the relationship between the topography of the PES and the thermodynamics. We review the superposition method here because it will be important later when we formulate the thermodynamic properties of a transformed energy surface, and because some additional developments need to be made before it can be applied to LJ55. The fundamental relation is

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\[ \Omega(E) = \sum_{E_i \leq E} g_s n_s \Omega_s(E) \]

where the prime indicates that the sum is now over an incomplete but representative sample of minima. The effect of \( g_s \) can be incorporated by using \( p_s(E') \), the probability of obtaining \( s \) in a minimization from a configuration generated by a microcanonical simulation at energy \( E' \). If the system is ergodic on the time scale of the simulation, then \( p_s(E') \approx g_s n_s \Omega_s(E') \). Hence,

\[ \Omega(E) \approx \sum_{E_i \leq E} p_s(E') \frac{\Omega_s(E)}{\Omega_s(E')} \]

Since we know the low energy limiting form of \( \Omega(E) \), where the contribution of the lowest energy minimum dominates, the proportionality constant in Eq. (4) can also be obtained.

This reweighting technique is analogous to histogram Monte Carlo, but instead of determining the configurational density of states from the canonical potential energy distribution, \( g_s \), effectively a density of minima, is found from the microcanonical probability distribution of being in the different basins of attraction. Hence there are some similarities with the microcanonical multihistogram approach of Calvo and Labastie.

The accuracy of the method depends on the statistical accuracy of \( p_s(E') \); this probability distribution needs to be obtained at an energy where all relevant minima are significantly sampled. This is possible for LJ55, however, for
standard thermodynamical formulas. The thermodynamic quantities can be calculated by the application of V~

figurational entropy associated with the icosahedral funnel is larger than for the fcc funnel. As a result of this entropy

vibrational frequency than the truncated octahedron, the con-

minima can both be sampled because of the large free energy barrier. Therefore, in the latter case the density of states is

obtained by adding two terms:

$$\Omega(E) = \Omega_{\text{fcc}}(E) + \Omega_{\text{rest}}(E).$$

(5)

The density of states of the fcc region of phase space, $\Omega_{\text{fcc}}(E)$, can be obtained by summing the density of states for all the low energy fcc minima using Eq. (2). (In fact the summation can be dispensed with since the term from the global minimum is the only term in $\Omega_{\text{fcc}}(E)$ that ever contributes significantly to $\Omega(E)$.) At the melting point an LJ38 cluster samples both the icosahedral and the liquid-like regions of configuration space and so Eq. (4) can be used to find $\Omega_{\text{rest}}(E)$. Once $\Omega(E)$ has been determined, various thermodynamic quantities can be calculated by the application of standard thermodynamical formulas. 40

The resulting equilibrium thermodynamic properties of LJ38 are shown in Fig. 5. The anharmonic expression for $\Omega$, was used with the anharmonicity parameters 51 determined by the technique given in Ref. 40. As there are many low energy icosahedral minima and as these minima have a lower mean vibrational frequency than the truncated octahedron, the configurational entropy associated with the icosahedral funnel is larger than for the fcc funnel. As a result of this entropy difference, there is a transition from the fcc to the icosahedral regions of configurational space at low temperature. The transition is centered at a temperature of $\sim 0.11$ eV and gives rise to the small peak in the heat capacity [Fig. 5(b)]. The finite-size analogue of the bulk melting transition occurs at $\sim 0.17$ eV, producing the main peak in the heat capacity. 52 These transitions are also reflected in the occupation probabilities for the different "phases" [Fig. 5(a)].

We can understand the effect these thermodynamic transitions have on global optimization by considering the result of cooling a cluster from the liquid-like state. The cluster is thermodynamically much more likely to enter the icosahedral funnel on quenching because the free energy associated with the icosahedral structures at the melting point is lower than that associated with the fcc structures. Furthermore, this effect is reinforced by the topology of the PES. The structure of simple atomic liquids has significant polytetrahedral character, 53,54 in contrast fcc structures have no polytetrahedral character, and icosahedra are somewhere in between. As a result of this greater structural similarity, the icosahedral funnel is more accessible from the liquid-like state. (A similar effect has been noted by Straley who found that crystallization from a simple liquid is much easier in a curved space, where a polytetrahedral packing is the global minimum, than in normal flat space where a close-packed crystal is most stable. 55,56) The combined effect of these two features is to make it extremely probable that on cooling the cluster will enter the icosahedral funnel. On further cooling, the cluster will remain trapped in this funnel even when the fcc structures become lower in free energy because of the large free energy barrier between the two funnels. Clearly this behavior will present difficulties for annealing methods, which simply try to follow the free energy global minimum down to zero temperature by gradual cooling.

The only opportunity to find the truncated octahedron by conventional simulation is at high temperature in the narrow window where the fcc structures have a small, but not yet insignificant, probability of being occupied and where the Boltzmann weights for the intermediate states, which mediate transitions between the fcc and icosahedral funnels, are large enough to make the free energy barrier surmountable. Indeed, we did observe the truncated octahedron in the microcanonical simulation ($T=0.185$ eV) from which we obtained the distribution of minima depicted in Fig. 3. However, minimization only led to this structure once in the entire 0.25 µs run. 57

Some thermodynamic properties of LJ38 are shown in Fig. 6. The partition function for this system was obtained previously in the development of the superposition method, 40 and it gives very good agreement with thermodynamic properties derived by more conventional means. 58 There is a single peak in the heat capacity curve corresponding to the melting transition at which the cluster passes from the Mackay icosahedron (perhaps with one or two defects) into the liquid-like state. For simulations in the middle of this melting region, the cluster passes back and forth between the different states. 59,60 Similarly, on cooling from the liquid the cluster enters the icosahedral funnel relatively easily.

FIG. 5. Equilibrium thermodynamic properties of the untransformed LJ38 PES in the canonical ensemble. (a) The probability of the cluster being in the fcc, icosahedral and liquid-like regions of bound configuration space. (b) The heat capacity, $C_\gamma$. The sample of 661 minima was obtained from 2500 quenches along a microcanonical MD run at $E=-150.1$ eV.
LJ$_{55}$’s single-funnel PES presents no thermodynamic obstacles to global optimization.

IV. GLOBAL OPTIMIZATION BY BASIN-HOPPING

The global optimization method that we analyze here belongs to the “hypersurface deformation” family of methods. In this approach the potential energy function is transformed in a way that is intended to make global optimization easier. The transformations usually try to lower the number of local minima—thus reducing the search space—and/or the barrier heights between minima—thus increasing the rates of transitions between minima. However, little attention is usually paid to the effects of the transformation on the thermodynamics of the system.

Once the global minimum of the transformed PES is found it is mapped back to the original surface in the hope that this will lead to the global minimum on the original PES. However, there is no guarantee that the global minima on the two surfaces are related and often there are good reasons to think that they are not. For example, one method suggested for clusters is to increase the range of the potential. Such a transformation can dramatically reduce the number of minima and also lower the barrier heights. However, it has been shown that changing the range of the potential for clusters can lead to many changes in the identity of the global minimum.

In contrast, the transformation that we apply to the PES is guaranteed to preserve the identity of the global minimum. The transformed potential energy $\tilde{E}_c$ is defined by

$$\tilde{E}_c(X) = \min\{E_c(X)\},$$

where $X$ represents the vector of nuclear coordinates and $\min$ signifies that an energy minimization is performed starting from $X$. Hence the energy at any point in configuration space is assigned to that of the local minimum obtained by the minimization, and the transformed PES consists of a set of plateaus or steps each corresponding to the basin of attraction surrounding a minimum on the original PES.

One can easily see that the transformation will have a significant effect on the dynamics. Transitions to a lower energy step are barrierless, and so relaxation down a funnel is like descending a multidimensional staircase. Of course, there are still uphill steps in energy to neighboring basins corresponding to higher energy minima, but the transition state regions have again been removed. Furthermore transitions can occur at any point along the boundary between basins of attraction, whereas transitions on the untransformed surface can occur only when the system passes along the transition state valley. As a result intrawell vibrational motion is removed and the system can hop directly between basins of attraction at each simulation step, hence our name for the algorithm, basin-hopping. In structural terms the system can now pass through large repulsive barriers on the untransformed PES. Similarly, if the method were applied to a polymer system it should remove entanglement effects because chains can pass through each other.

However, the transformation does not entirely remove the energy barriers between two funnels but only the component due to transition states. Therefore, it is not self-evident that it will aid global optimization on multiple-funnel PES’s; this effect depends on how the transformation affects the thermodynamics of the system.

To explore the transformed PES we simply use a canonical Monte Carlo simulation. For clusters we also have to consider how to restrict our search of configuration space to that for bound clusters. This problem is more pressing than for the original PES, since the transformation has removed many of the energy barriers to the dissociation of the cluster. We have considered two approaches to achieve this. The first
is to place the cluster in a tight-fitting box; the second is to reset the coordinates at each successful step to the relevant minimum. In the latter case the method is equivalent to Li and Scheraga’s Monte Carlo Minimization approach, although they did not conceive of the method in terms of a transformation of the PES. White and Mayne have now shown that resetting the coordinates is probably the best strategy, and it is the one we concentrate on here.

We have found that the basin-hopping algorithm performs well. In our application to Lennard-Jones clusters it found all the known lowest energy configurations up to 110 atoms, including those with non-icosahedral global minima. Furthermore, it has performed impressively for a wide range of cluster systems, some of which have a considerably more rugged PES than Lennard-Jones clusters.

In Fig. 8 we show some basin-hopping trajectories for LJ$_{38}$ and LJ$_{55}$. It can be seen that the cluster is able to pass between the fcc and icosahedral funnels of LJ$_{38}$ over a range of temperature. Therefore, the free energy barriers must be lowered by the transformation. In the next section we will examine the thermodynamics of the transformed PES to discover the basis for this change. The predicted acceleration of the dynamics is also evident from the frequency of transitions between the different states of LJ$_{55}$ and LJ$_{38}$ (Fig. 8). The timescale (in numbers of steps) for these processes is much more rapid than for similar processes on the original PES, where even diffusion across a flat free energy landscape can be slow. However, it should be remembered that a single MC step on $\bar{E}_c$ is much more computationally expensive than an MD step or an MC cycle on the untransformed PES, because it involves a minimization.

A. Thermodynamics for $\bar{E}_c$

To calculate the thermodynamics for the transformed energy surface, $\bar{E}_c$, the partition of configuration space that we used in Sec. III [Eq. (2)] is appropriate since the plateaus of $\bar{E}_c$ correspond to the basins of attraction on the original PES. However, we need to derive the form of $\Omega_s(E)$. The configurational density of states of a plateau $s$, $\Omega_{c,s}(\bar{E}_c)$ is simply

$$\Omega_{c,s}(\bar{E}_c) = \delta(\bar{E}_c - E_s) A_s ,$$

(7)

where $A_s$ is the hyperarea of the basin of attraction of minimum $s$. This expression is then convoluted with the kinetic density of states to give

$$\Omega_s(E) = \frac{2 \pi m}{\Gamma(\kappa/2) h^\kappa} A_s (E - E_s)^{\kappa/2 - 1} \theta(E - E_s) ,$$

(8)

where $m$ is the mass of an atom, $h$ is Planck’s constant, $\kappa$, the number of vibrational degrees of freedom, is $3N - 6$ and $\theta$ is the Heaviside step function. As $A_s$ cannot be readily calculated, we cannot use Eq. (2) to obtain the total density of states but instead employ Eq. (4). For $\bar{E}_c$, the latter approach can even be applied to LJ$_{38}$ since, as Fig. 8(b) shows, conditions can be found where the fcc, icosahedral and liquid-like regions of configurational space are all significantly sampled. However, since we do our simulations on $\bar{E}_c$ using standard Metropolis Monte Carlo sampling, we actually use the canonical analogue of Eq. (4):

$$Z(\beta) \propto \sum \lambda p_\lambda(\beta') \frac{Z_s(\lambda)}{Z_s(\beta')} ,$$

(9)

where $Z$ is the partition function, the inverse temperature $\beta = 1/kT$ and the probability distribution was obtained from a basin-hopping trajectory at $\beta'$. To use this equation, we must first Laplace transform the expression for $\Omega_s$ in Eq. (8) to give $Z_s$, the partition function of a plateau:

$$Z_s(\beta) = \frac{2 \pi m}{\beta h^2} A_s e^{-\beta E_s} .$$

(10)

Equation (9) then reduces to
\[ Z(\beta) \propto \sum_s p_s(\beta') e^{-E_s(\beta'-\beta)}. \]  \hspace{1cm} (11)

It is interesting to note that this equation is identical to that used in histogram MC.\textsuperscript{41,48,49} This equivalence stems from the fact that the potential energy is constant on each plateau \( s \). Hence the multihistogram techniques, in which information from a number of runs at different temperatures is used to construct the partition function, could also be applied. However, such an approach was not necessary in this study.

The \( A_s \) values depend upon how configuration space is restricted to bound clusters. When resetting the coordinates to those of the minimum at each successful step, configuration space is restricted to a hypersphere (with radius corresponding to the maximum step size) around each local minimum, and so the thermodynamics (and the performance of the basin-hopping algorithm\textsuperscript{75}) is somewhat dependent on the maximum step size used. When using a container the total accessible configuration space, \( \Sigma n_s A_s \), is the volume of a \( \kappa \)-dimensional hypersphere whose radius corresponds to that of the container. Therefore, the thermodynamic contribution of irrelevant regions of configuration space corresponding to clusters of low density, or with atoms evaporated, increases with container size. Hence, it is advantageous to use a tight-fitting container, and for this situation the thermodynamics are similar to when the coordinates are reset.

One further consideration is that one can imagine situations where resetting the coordinates breaks detailed balance. For two minima A and B which are adjacent on the PES, it may be that no part of the basin of attraction of minimum B is within the maximum step size of minimum A, whereas the basin of attraction of A lies within the maximum step size of minimum B. If so the system can pass from B to A but never from A to B, thus breaking the detailed balance condition. Although this means that the "resetting" implementation of the basin-hopping algorithm might not formally produce a canonical ensemble, in practice it does to a good approximation—the thermodynamic properties calculated using Eq. (11) for samples generated at different temperatures are in reasonable agreement.

For the transformation of the LJ\textsubscript{38} PES to reduce the free energy barriers between the funnels, the probability of the system occupying the intermediate states between them must be non-negligible under conditions where the icosahedral and fcc structures also have significant occupation probabilities. The thermodynamics of the transformed PES have just these properties (Fig. 9).\textsuperscript{74} The transitions have been smeared out and there is now only one broad peak in the heat capacity. Most significantly, the probability that the system is in the basin of attraction of the global minimum decays much more slowly, and the temperature range for which both the liquid-like minima and the global minimum have significant probabilities is large. Basin-hopping runs anywhere in this temperature range are able to locate the global minimum rapidly from a random starting point. At the lower temperatures in this range, e.g., \( T = 0.4 \, \text{ek}^{-1} \), there is still a small free energy barrier and so the cluster can be trapped in one of the funnels for many steps [Fig. 8(a)]. However, at higher temperatures the system passes more rapidly between the states [Figs. 8(b) and 8(c)].

In this way we can correlate the performance of the basin-hopping algorithm with the thermodynamics. The temperature dependence of the first-passage time for reaching the global minimum from a random configuration is shown in Fig. 10. As the temperature decreases, the probability of the system being in the high energy intermediate states between the two systems decreases. The resulting increasing free energy barrier causes the first-passage time to rise at low temperatures. Interestingly, and perhaps unexpectedly, there is little rise at higher temperature. For many systems, e.g., proteins,\textsuperscript{75,76} the first-passage time has a minimum as a function of temperature; the rise at high temperature is because the equilibrium probability of being in the global minimum tends to zero. However, for the basin-hopping algorithm, the first-passage time at high temperature is approximately constant.\textsuperscript{77} In fact the probability of being in the global minimum never goes to zero, e.g., \( p_{A_h}(T = \infty) = 0.0024 \).

The thermodynamics on the transformed surface for LJ\textsubscript{55} are also considerably broadened and the transitions have been smeared out over a large temperature range (Fig. 11). Consistent with this, there is a greater probability that one of the states intermediate between the global minimum and the
liquid is occupied. As expected for the single-funnel topography of LJ$_{55}$, the basin-hopping algorithm needs far fewer steps to find the global minimum than for LJ$_{38}$. It is remarkable that for a wide range of temperature ($T > 1.5 e k^{-1}$), the method on average requires fewer than 200 minimizations to find the global minimum from a random starting point (Fig. 12) despite the estimated $10^{21}$ minima on the PES. This is a testament not just to the efficiency of the basin-hopping algorithm but also to the dramatic effect that a funnel has in guiding the system towards the global minimum. As with LJ$_{55}$ the first-passage time is approximately constant at high temperature (Fig. 12); $p_{f}(T = \infty) = 0.11$.

The superposition method allows us to connect the thermodynamics to the properties of the PES. Hence we can understand the thermodynamics of the transformed surface by examining expressions for the canonical probability that the system is in a minimum or on a plateau $s$. On the untransformed PES

$$p_s(\beta) = \frac{n_s \exp(-\beta E_s)}{\bar{\nu}_s^N} \sum_s n_s \exp(-\beta E_s),$$

(12)

where $\bar{\nu}_s$ is the geometric mean vibrational frequency of minimum $s$, and we have used the harmonic approximation—the more accurate anharmonic form is complicated and does not provide any additional physical insight. For the transformed PES,

$$p_s(\beta) = \frac{n_s A_s \exp(-\beta E_s)}{\sum_s n_s A_s \exp(-\beta E_s)},$$

(13)

In Eqs. (12) and (13) the sums are over all the minima on the potential energy surface.

The two expressions (12) and (13) differ only in the vibrational frequency and $A_s$ terms. The fcc to icosahedral and the icosahedral to liquid transitions are caused by the greater number of minima (and therefore the larger entropy) associated with the higher temperature states. On the untransformed surface this effect is reinforced by the decrease in the mean vibrational frequencies with increasing potential energy [Fig. 13(a)]. Although the dependence of $\bar{\nu}_s$ on $E_s$ may seem relatively small, because $\bar{\nu}_s$ is raised to the power $3N - 6$ it leads to a significant increase in the entropy of the higher energy states, sharpening the transitions and lowering
the temperature at which they occur. In contrast, $A_s$ decreases rapidly with increasing potential energy [Fig. 13(b)]. The decrease in $A_s$ reduces the entropy of the higher energy states, causing the transitions to be broadened and the temperature at which they occur to increase.

To obtain the $A_s$ values shown in Fig. 13(b) we used $p_s(\beta')$, the probability with which a plateau $s$ was visited in a basin-hopping trajectory. As $p_s(\beta) \propto g_s n_s A_s \exp(-\beta E_s)$, relative values of $g_s A_s$ can be found. A similar inversion of $p_s(E')$ for the untransformed surface allows values of $g_s$ to be found. However, the $g_s$ and $g_s A_s$ values are for different sets of minima. The two samples of minima overlap completely for the low energy minima, and so $A_s$ values for these minima can be calculated directly. For higher potential energies, where the spectrum of minima is quasi-continuous, we find the average value of $A_s$ in an interval of the potential energy, $\Delta E_c$, from $\Sigma s g_s A_s / \Sigma s g_s$ where the sums are only over the minima in the two samples which occur in that energy interval.

The superposition approach also allows us to comment on the thermodynamics of a related optimization algorithm which steps directly between connected minima on the PES (the steps are achieved using a transition state searching algorithm). As with the basin-hopping algorithm the detrimental effects of vibrational motion are thus removed. This method has been successfully used to find low energy structures of amorphous semiconductors and its performance for a few Lennard-Jones clusters has also been illustrated. If each step is accepted with a Metropolis criterion then it is easy to formulate the thermodynamics for this discrete space of minima: $p_s(\beta) = \frac{n_s \exp(-\beta E_s)}{\Sigma n_s \exp(-\beta E_s)}$. This expression is intermediate between Eqs. (12) and (13).

V. CONCLUSIONS

We can now explain in detail why the basin-hopping or Monte Carlo Minimization method is successful. First, the staircase transformation removes the transition state barriers between minima on the PES without changing the identity of the global minimum, accelerating the dynamics. Second, it changes the thermodynamics of the surface, broadening the transitions so that even for a multiple-funnel surface such as that of LJ$_{38}$, the global minimum has a significant probability of occupation at temperatures where the free energy barrier for passage between the funnels is surmountable.

It is this latter feature which is especially important in overcoming the difficulties associated with multiple funnels and represents a new criterion for designing successful global optimization methods. Most previous hypersurface deformation schemes have been developed without regard to the thermodynamic effects of the transformation and so, in some cases, they may make optimization more difficult. The use of Tsallis statistics to improve annealing algorithms is another example of how thermodynamic insight can be used.

The broadened transition also means that the global minimum can be found by simulations over a relatively broad range of temperature. Therefore, simple canonical Monte Carlo is sufficient to search the transformed surface and the performance is relatively insensitive to the choice of temperature. However, any of the more advanced techniques for searching PES’s such as simulated annealing, genetic algorithms, jump walking, Tsallis statistics, and simulated tempering could be used to search the transformed PES. As many of these methods have been designed to work on untransformed PES’s it is not obvious that they will improve performance, and preliminary investigations have not yet indicated that any of them provides a significant advantage. It is also interesting to note that the most successful applications of genetic algorithms to clusters refine each new configuration generated by mutation or mating by minimization; this minimization step has been shown to be crucial. These genetic algorithms are therefore searching
the transformed PES, $E_\text{r}$, and it is probably this feature which is responsible for their success. In fact, we have succeeded in obtaining comparable results for Lennard-Jones clusters with a modified genetic algorithm. The most efficient method is likely to be system-dependent and draw upon features from a number of different approaches.

$L_1^{3h}$ could prove to be an interesting example to explore the use of other techniques to search the transformed surface because its global minimum, even with the basin-hopping algorithm, is difficult to find. The free energy barriers between the decahedral and fcc funnels, although reduced by the transformation of the PES, are still large enough to make global optimization difficult. Therefore, $L_1^{3h}$ may provide a suitable system in which to investigate algorithms that enhance barrier crossing or simulate an ensemble which has higher probability distributions than the canonical ensemble.

The broadened transition that results from the staircase transformation also differs markedly from the optimum conditions for protein folding, if we assume that proteins have evolved single-funnel surfaces in order to fold efficiently. A steep funnel provides a large thermodynamic driving force for relaxation to the global minimum, and also leads to a sharp thermodynamic transition. However, in global optimizations one cannot make assumptions about the topography of the PES, and on a multiple-funnel PES features such as steepness can exacerbate the difficulties associated with trapping in secondary funnels.

Moreover, in protein folding there is the extra requirement that the folded protein must remain localized in the native state, and this necessitates a sharp transition. There is no need for a global optimization method to mimic this property. Indeed, when applied to a PES with multiple funnels the optimum temperature for the basin-hopping approach is above the transition, where the system only spends a minority of its time associated with the global potential energy minimum.

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26 An up-to-date list of the Lennard-Jones global minima, along with points of each global minimum, is maintained at The Cambridge Cluster Database, URL http://brian.ch.cam.ac.uk/CCD.html.
31 R. H. Leary (personal communication).
32 D. J. Wales and A. R. Markham (unpublished).
36 The pathway shown here is considerably lower in energy than the one reported in Ref. 33 and was the result of a more extensive search in which over 7000 minima and 10000 transition states were found.
37 This suggestion has been confirmed in simulations using umbrella sampling to characterize the free energy barrier. J. P. K. Doye and D. J. Wales, eprint cond-mat/9808265 at http://xxx.lanl.gov.
51 The values of the anharmonicity parameters were: $a_0=0.39$, $a_\text{anis}=0.43$ and $a_\text{isot}=0.40$. The shape of the heat capacity curve and the position of the transitions are slightly sensitive to the values used. These values are an improvement on those used in Ref. 14.
52 It should be noted that the melting transition does not give rise to a van der Waals loop in the microcanonical caloric curve. The presence of a loop in the caloric curve obtained by Calvo using a multihistogram Monte Carlo approach (see Ref. 90) probably results from the cluster being trapped in the fcc funnel up until melting.
57 The conversion of the time from reduced units, $(m^2T^2)^{1/2}$, into seconds is based on the values of $\epsilon$, $m$ and $\sigma$ appropriate for argon.
Chap. 3. The thesis can be accessed on the web at URL http://brian.ch.cam.ac.uk/~jon/pub.html.

65 Reference 58, Chap. 6.
74 The temperature dependence of the first-passage time is considerably different if the maximum step size is dynamically adjusted to give a constant acceptance ratio. With this approach the first-passage time has a minimum, because the maximum step size increases at higher temperature, changing the thermodynamics and decreasing the equilibrium probability of being in the global minimum.
77 The equilibrium thermodynamic properties of the transformed LJ$_{75}$ and LJ$_{55}$ PES’s reported here are slightly different from Ref. 14 because of the dependence of the thermodynamics on the maximum step size. In Ref. 14 the average maximum step size was 0.38$\sigma$ for LJ$_{38}$ and 0.41$\sigma$ for LJ$_{55}$.
79 However, depending on the algorithm used to locate transition states, detailed balance may not be obeyed, and so the method may not formally simulate a canonical ensemble in the space of minima. In our experience with using the eigenvector-following technique to locate transition states, instances have been found where when we start from minimum A we can find a transition state that connects it to minimum B, but when we start from minimum B this transition state is never found, thus breaking detailed balance.